Pergamon

Int. J. Solids Structures Vol. 34, No. 18, pp. 2303–2320, 1997

0020–7683/97 $17.00 + .00

PII: S0020–7683(96)00168–0

A NEW THEORY FOR FREE-SURFACE FORMATION IN SOLID CONTINUA

M. M. RASHID
Department of Civil and Environmental Engineering, University of California at Davis, Davis, CA 95616-5294, U.S.A.

(*Received* 18 *July* 1995; *in revised form* 26 *July* 1996)

Abstract—In essence, fracture mechanics consist of the conventional boundary-value problem formulation of continuum mechanics, along with a variety of fracture criteria that govern the advance of crack fronts. The assumption that the material response is governed by a local constitutive model everywhere in the body, even at points that are arbitrarily close to the crack front, generally leads to an unbounded stress field as the crack front is approached. Also, processes such as metal-cutting and penetration share with fracture the essential feature of new-free-surface formation, but do not fit easily within the conventional fracture theory. For these and other reasons, it seems worthwhile to seek a broader theoretical construct which encompasses surface separation in a more general setting. With this motivation, a new theory is proposed which applies generally to surface separation in solid continua, but which nonetheless yields fracture-mechanics-type predictions in appropriate special cases. The proposed *exclusion region theory* involves identification of a small material neighborhood that contains the separation front. A generalized constitutive description that derives directly from the local constitutive model is constructed for the exclusion region. A *separation criterion* is formulated with reference to tractions on, and/or distortion of, the exclusion region. The direction-of-advance of the separation front is determined as a natural consequence of the separation criterion. The material parameters appearing in the separation criterion can generally be determined from conventional fracture tests. The theory has been implemented in a finite element code. Two example problems illustrating certain important aspects of the theory are presented. © 1997 Elsevier Science Ltd.

1. INTRODUCTION AND BACKGROUND

A fundamental hypothesis of classical continuum mechanics is the time-invariance of the connectivity properties of material regions. More precisely, material curves, surfaces, and volumes that are subsets of a continuum body, are presumed to preserve their topological connectedness as the body deforms. An immediate consequence of this hypothesis is that all material points in a continuum body belong to exactly one of two time-invariant sets: the set of interior points, and the set of boundary points. However, common experience indicates that, in real bodies, membership in these two disjoint sets is not always time-invariant: it is quite possible for an interior surface to separate into a pair of boundary surfaces. The subdiscipline of fracture mechanics has evolved in response to the need to model such processes. The development of fracture mechanics has focused rather strongly on formulating criteria that determine *whether or not* a sharp crack will advance, and, to a somewhat lesser extent, on constructing models for the *direction* of crack advance. In fracture mechanics, predictive models for the initiation of crack advance, and for subsequent crack-path direction, are rather closely linked to the particular constitutive model used in the boundary-value problem for the cracked body. By far the most extensive body of results has been compiled for linearly elastic material behavior.

This paper represents an attempt to formulate a continuum-mechanical theory that admits free-surface formation in a rather general context. The work is motivated by a desire to separate, to the extent possible, modeling considerations of the bulk, smoothly-deforming continuum, from those of the surface-separation phenomenon itself. The goal is a theoretical framework that can accommodate arbitrary constitutive characterizations of surface separation, in much the same way that classical continuum mechanics accommodates arbitrary local constitutive relations for the bulk response.

Although the proposed theoretical framework is intended to encompass arbitrary inelastic material behavior and finite deformations, it is herein made specific to the special case of planar deformations of linearly elastic bodies. This is done for two reasons: first, this special case affords considerable simplicity and clarity with regard to exposition of the underlying structure and assumptions of the theory. Second, specialization of the proposed theory to linear elasticity allows for comparison with, and establishment of correspondence with, linear elastic fracture mechanics (LEFM). This correspondence is discussed in Section 3.2, after exposition of the theory. It bears emphasis that the real utility of the proposed theory, here named the "Exclusion Region" (ER) theory, is to be found in applications to finitely-deforming inelastic materials. In the course of the development, it will become evident that the particular material model used to describe the bulk-continuum material behavior engenders no restrictions in the formulation of the ER theory.

Kinematically, free-surface formation in continua is a consequence of the propagation of a *singular curve* (or *separation front*) Γ through the body (Fig. 1). The velocity field is singular on Γ, in the sense that different limiting values are obtained as Γ is approached from different directions. The interior material surface described by the motion of Γ separates into a pair of boundary surfaces, which are then subject to boundary conditions. The deformation gradient may or may not become unbounded as Γ is approached, but in any case is singular on Γ, in the sense that no single-valued limit can exist there. Accordingly, any material volume through which Γ passes, however small, does not suffer a nearly uniform deformation field. Although this observation seems obvious, it has serious implications with regard to the modeling of constitutive behavior in the presence of free-surface formation. Specifically, the physical relevance of any local constitutive model relies upon the continuity of the deformation field. In other words, given an arbitrary point in a body, there must exist a (perhaps very small) material neighborhood containing this point, the boundary of which suffers an "essentially uniform" deformation. This fundamental hypothesis underlies the physical relevance of all local constitutive models. Points that lie on the path of Γ violate this hypothesis; this fact suggests that modeling free-surface formation requires an enlarged conception of the constitutive model.

A local constitutive model may be regarded as simply a means of associating a traction distribution on the boundary of a material neighborhood with the (history of) displacements on this boundary, in the case where these boundary displacements are linearly related to a reference position. This boundary traction–boundary displacement relationship is scale-invariant, as long as the boundary displacements are restricted to derive from a constant displacement gradient. Accordingly, the material neighborhood can be taken to be arbitrarily small: this leads to the pointwise equivalence of the "material stress" (i.e., the tensor that specifies the boundary tractions from the constitutive model), and the "mechanical stress" (i.e., the tensor that is related to the tractions through $\mathbf{t} = \mathbf{Tn}$ at every point in the body). The local constitutive model thereby facilitates the link between the motion of the body and the differential equations of motion.

This somewhat "nontraditional" perspective is familiar in micromechanics and homogenization theory. It is adopted here with the intention of illuminating the basic assumptions of classical continuum mechanics that require modification in the presence of free-surface

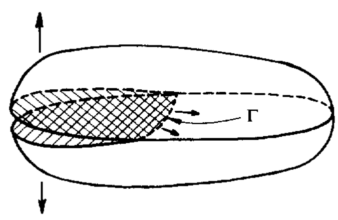

Fig. 1. Schematic of a solid body with a propagating singular curve Γ, leaving in its wake a pair of newly-created free surfaces.

formation. Specifically, because any material neighborhood containing a separation front does not suffer a linear displacement distribution on its boundary, the constitutive model must be enlarged to accept as "input" some other, more general displacement distribution. However, any departure from the linear-displacement-distribution assumption leads to a loss of scale-invariance of the resulting constitutive model. This length-scale dependence must be accepted, and indeed is physically appropriate in the context of free-surface formation.

Based on the foregoing reasoning, a brief summary of the exclusion region theory may be sketched, as follows. A small (but finite) neighborhood containing the separation front is identified, whose boundary is assumed to suffer an arbitrary deformation. A displacement field is postulated within this "exclusion region," from which the traction distribution on the boundary is obtained by evaluating the local constitutive model on the boundary. The enlarged constitutive description of the exclusion region thereby derives directly from the local material model, whatever it may be. Advance of the separation front, and the direction of advance, are derived from a "separation function" which, in turn, is related to tractions on, and/or distortions of, the exclusion region boundary.

In conventional fracture mechanics theory, the need for some type of broadened constitutive formalism is not directly addressed. Instead, a conventional boundary-value problem is solved, with the result that the stress field is usually unbounded at the crack tip. Of course, it is well understood that the unbounded field ceases to be physically meaningful within some neighborhood of the crack tip. The singular field is still useful, however, for the purpose of characterizing the loading environment close to the tip. Aspects of the singular stress field may therefore be correlated with the propensity of the crack to advance. This is the fundamental insight of Irwin's (1957) stress-intensity-factor approach to fracture in linearly elastic bodies. The asymptotic behavior of the stress field has been elucidated for other material models as well—e.g., the deformation theory of elastoplasticity (Hutchinson, 1968; Rice and Rosengren, 1968), linear viscoelasticity (Willis, 1967), and nonlinearly viscous materials (Sharma *et al.*, 1994). Some results are also available for hyperelastic solids suffering finite deformations (Knowles and Sternberg, 1973; Duva, 1990). In all cases, a predictive model for crack advance can be constructed by attempting to correlate a singularity-strength parameter that appears in the asymptotic solution, with experimentally-observed specimen load levels at which crack advance commences. This process requires that the relevant asymptotic solution be available, and it results in fracture criteria that are closely linked to the details of this asymptotic solution—and therefore to the material model to which the solution corresponds. Also, some recent numerical studies have suggested that the region of dominance of some inelastic asymptotic solutions is so small that such solutions are not very useful for the purpose of characterizing fracture behavior—see the review by Liebowitz *et al.*, (1995).

Within certain contexts, theorems are available that establish a correspondence between a strain-energy release rate, and certain path-independent integrals taken over paths that encircle the crack front (Eshelby, 1956; Rice, 1968). Use of J and other path integrals to characterize the conditions at the tip of a crack has been the subject of intensive development effort and interest since the late 1960's (e.g., Freund, 1978; Liu *et al.*, 1981; Moran and Shih, 1987). The usefulness of such path-independent integrals follows from the physical postulate that a crack will advance when it is energetically favorable for it to do so. For problems in which the requirements for path-independence are satisfied, this approach has proven to be very useful. For example, it alleviates the need to resolve with great precision the asymptotic stress field near the separation front. In a computational setting, this feature is of great practical significance. Also, the applicability and limitations of the J-integral approach with regard to determining the direction of crack advance under mixed-mode conditions has been explored (Herrmann and Herrmann, 1981).

The original ideas of Barenblatt (1962) and Dugdale (1960) have been used extensively and in many forms to model crack advance, especially with a view toward numerical simulation. Barenblatt (1962) hypothesized the existence of near-tip closing tractions that vary smoothly with relative displacement of the crack faces, and which vanish for sufficiently large separation. If the traction profile is chosen properly, then the stress state in the surrounding elastic medium is rendered nonsingular. The motivation for Barenblatt's original work stemmed from interatomic forces acting at finite distances across cracks in brittle solids. On the other hand, Dugdale (1960), motivated by near-tip yielding in ductile metals, proposed a yielding-strip model wherein the stress state in a thin strip ahead of the crack tip is one of uniaxial yield. Many more recent modeling efforts have involved a combination of the two concepts proposed by Barenblatt and Dugdale, usually with the intention of modeling the behavior of a thin region of material ahead of the crack tip as it fails (Fager *et al.*, 1991). For example, Knauss and Losi (1993) considered a thin strip of nonlinearly viscoelastic, voiding material in a linearly viscoelastic medium. The stress state in the nonlinear strip is assumed to be uniaxial, as in the Dugdale model. This work emphasizes the role of rate effects in crack propagation in viscoelastic media. In other work (Hillerborg *et al.*, 1976; Liaw *et al.*, 1990), fracture in concrete has been modeled using simple crack-tip-closing traction profiles that are limited by the tensile strength of the concrete. Finite element analyses are reported in both of these studies, wherein the crack path is assumed known *a priori* and the crack-tip-closing tractions are applied across element boundaries. More recently, Tvergaard and Hutchinson (1992) have carried out a detailed finite element study of normal-mode crack initiation and growth in rate-independent elastic-plastic materials. In their work, a trilinear crack-tip-closing traction profile is used, and the surrounding medium is modeled by the J_2 flow theory of plasticity with nonlinear hardening. A large-deformation Lagrangian finite element formulation is used. The work of Tvergaard and Hutchinson (1992) is focused largely on the evolution of certain fracture parameters, such as apparent toughness and a process-zone length, during the early stages of crack advance.

In addition to these efforts to introduce an additional constitutive structure to model surface separation, a related computational procedure has recently emerged, wherein a finite-size process zone on the crack path is modeled using a local damage model. Specifically, Xia and Shih (1995) and Xia *et al.* (1995) consider mode-I crack growth in elastic-plastic solids with a number of different geometries. The essence of their formulation is a single row of finite elements along the crack plane, in which a modified-Gurson-type local damage model is presumed. In the remainder of the domain, a standard J_2-flow-theory plasticity model is applied. The results of such computations are necessarily strongly dependent on the thickness of the layer of damaging elements; this dimension is retained as an empirical parameter in this formulation. When a scalar damage parameter reaches a prescribed critical value at a given integration point, the associated element is rendered extinct over a prescribed number of subsequent load steps. In this way, the formation of new free surfaces is approximately modeled in this computational setting. This approach appears to be well-suited to the study of model problems involving monotonic, symmetric loading. Further work is needed to establish the applicability of some modified version of this approach to problems wherein the crack path cannot be inferred from symmetry.

The formulation of the proposed exclusion region theory is independent of any computational formulation; however, the theory admits a convenient finite element implementation. Aspects of this implementation are outlined in Section 3.1. To illustrate the application of the theory, the general construct is made specific to a special case in the next section: small, quasistatic, planar deformations of linearly elastic bodies. In the ER theory, however, modeling of surface separation is decoupled from the constitutive characterization of the bulk continuum, in the sense that the *form* of the separation criterion does not depend on an asymptotic solution of the boundary-value problem. Within the confined context of linearly elastic material response, the proposed exclusion region theory renders predictions that are equivalent to linear elastic fracture mechanics (LEFM). A numerical example is presented in Section 3 that demonstrates this fact. Whereas this "baseline" correspondence with a familiar theory is a crucial first step in establishing the physical relevance of the new theory, the real strength of the exclusion region formulation is expected to lie in its *lack* of inherent restrictions regarding the type of material response, geometry, etc.

Notation is standard throughout, with boldface symbols representing vectors and rank-two tensors, and lightface symbols standing for scalars and tensorial components.

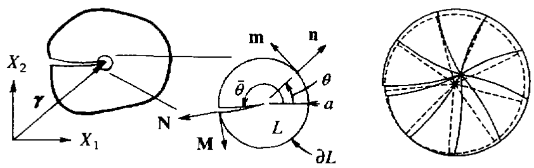

Fig. 2. Exclusion region in a two-dimensional domain (left), notational conventions (center), and an illustration of the assumed displacement field (right).

2. THE EXCLUSION REGION THEORY

2.1. *Displacement field and equilibrium*

In the present context of small deformations and planar geometry, the spatial curve Γ (i.e., the separation front) becomes a point within the domain B occupied by the body ($B \subseteq R^2$), and has position vector γ (Fig. 2). A polar coordinate system $\{r, \theta\}$ is established at Γ, with θ measured counterclockwise from the global x_1-direction. Points of ∂L are described by $\gamma + a\cos\theta\,\mathbf{e}_1 + a\sin\theta\,\mathbf{e}_2$, where $\mathbf{e}_i$ is the unit vector in the x_i-coordinate direction. The outward unit normal to ∂L is defined by $\mathbf{n} = (\cos\theta, \sin\theta)$, its in-plane normal by $\mathbf{m} = (-\sin\theta, \cos\theta)$, and the traction vector on ∂L by $\mathbf{t} = \mathbf{Tn}$, where $\mathbf{T}$ is the Cauchy stress tensor. The newly-formed pair of free surfaces in L is described by $\theta = \bar{\theta}(r)$. If the new free surfaces lie along a radial line within L, then $\bar{\theta}(r)$ is simply a constant. (There is no difficulty involved in abandoning the single-valued function $\bar{\theta}(r)$ in favor of a more general description of the newly-formed free surfaces within L. However, because the radius of the ER is small, this simple representation is expected to be entirely adequate for most purposes.)

The first objective is to construct a generalized constitutive model for the exclusion region L. It is desired that this generalized constitutive model be derivable from the local constitutive model that applies to the bulk continuum. Also, the constitutive relation for L should be consistent with the local material model, in the sense that the generalized model should reduce to the local one in the absence of surface separation.

As was discussed in the Introduction, a mechanical constitutive model may be regarded as a *functional* $F\{\cdot\}$ that relates the displacement (or its history) $\mathbf{u}$ on the boundary of a material neighborhood V, to the traction $\mathbf{t}$ on the boundary ∂V:

$$\mathbf{t}(\mathbf{x}) = F\{\mathbf{u}(\cdot)\}, \quad \mathbf{x} \in \partial V. \tag{1}$$

A *local* constitutive model is obtained if the argument of $F\{\cdot\}$ in (1) is restricted to be consistent with a linear, but otherwise arbitrary, spatial variation of displacement, i.e., such that

$$\mathbf{u}(\mathbf{x}) = \mathbf{u}^0 + \mathbf{H}(\mathbf{x} - \mathbf{x}^0), \tag{2}$$

where $\mathbf{H}$ is a constant, rank-two tensor (and small in the geometrically linear theory). Under this restriction, $F\{\cdot\}$ is scale-invariant for small V. Accordingly, as the size of V approaches zero, the *functional* $F\{\cdot\}$ in (1) can be replaced by a *function* of the displacement gradient (or its history), whereas the traction distribution on ∂V can be replaced by the Cauchy stress tensor. The resulting constitutive function is valid at each material point.

Because the exclusion region L contains a separation front, the displacement on ∂L is not adequately represented by a linear function in $\mathbf{x}$. Instead, the domain of the functional $F\{\cdot\}$ must be expanded to include more general displacement distributions on ∂L, including, in particular, those that suffer a jump at the new free surfaces. However, as the restriction to linear displacement distributions is abandoned, so goes the scale-invariance of the resulting constitutive model: no longer will it be possible to obtain a pointwise material characterization by passing to the limit of a vanishingly small material neighborhood. For this reason, the size of L, as given by the radius a, becomes a parameter in the constitutive model.

As mentioned above, in the present context it is desirable to seek a generalized constitutive model for the exclusion region that derives directly from the local material model that applies outside the ER. To this end, it seems reasonable to postulate that material points *on the boundary* ∂L of the ER obey the *local* material relation. This postulate allows for the determination of the traction distribution on ∂L, provided that a displacement gradient on ∂L can be obtained for use in the local material model. The displacement gradient on ∂L is available if a displacement field is postulated *everywhere* in L. There are other reasons, as well, for postulating a displacement field in L as part of the constitutive model for L: first, as the separation front advances, the exclusion region moves with it. Therefore, material points that are currently outside L may previously have been inside L. If the local constitutive model is history-dependent, consistency with the local material relation requires that the strain histories of such material points must be available. Secondly, in applying the proposed model to dynamic problems, the momentum of the ER can be determined only if the velocity field within it is known.

At issue, then, is the construction of a displacement field $\mathbf{u}(r, \theta)$ within L: this is the key ingredient in the generalized constitutive model for the exclusion region. To this end, the following observations and requirements are noted:

(1) $\mathbf{u}(r, \theta)$ must suffer a jump across the pair of new free surfaces within L; i.e., at $\theta = \bar{\theta}(r)$.
(2) $\mathbf{u}(r, \theta)$ must match the (arbitrary) displacement distribution on ∂L; i.e., there must hold $\mathbf{u}(a, \theta) = \hat{\mathbf{u}}(\theta)$, where $\hat{\mathbf{u}}(\theta)$ is the displacement on ∂L.
(3) The displacement must be single-valued at $r = 0$, i.e., at the separation front.
(4) $\mathbf{u}(r, \theta)$ must lead to a traction distribution $\mathbf{t}(\theta)$ on ∂L that renders the exclusion region in overall force- and moment-equilibrium, regardless of the boundary displacement $\hat{\mathbf{u}}(\theta)$. It is worth observing that questions of equilibrium (or momentum balance) do not arise in *local* constitutive theory, because scale-invariance implies that each infinitesimal material region suffers a traction distribution that corresponds to a constant stress, which always satisfies equilibrium.
(5) It is desirable that $\mathbf{u}(r, \theta)$ be as simple as possible in form.

With this guidance, the assumed form from the displacement field within L is taken to be

$$\begin{aligned}
\mathbf{u}(r, \theta) &= \frac{r}{a}[\hat{\mathbf{u}}(\theta) + \mathbf{w}P(r, \theta)] + \left(1 - \frac{r}{a}\right)\mathbf{g} - \frac{r}{a}\left(1 - \frac{r}{a}\right)v\,\mathbf{m}(\theta), \\
P(r, \theta) &= H(\theta - \bar{\theta}(a)) - 1 + \frac{\bar{\theta}(r) - \bar{\theta}(a)}{2\pi}, \quad 0 \leqslant r < a, \quad \bar{\theta}(r) \leqslant \theta < \bar{\theta}(r) + 2\pi, \\
\mathbf{w} &= \hat{\mathbf{u}}(\bar{\theta}(a)) - \hat{\mathbf{u}}(\bar{\theta}(a) + 2\pi).
\end{aligned} \tag{3}$$

In eqn (3), $\hat{\mathbf{u}}(\theta)$ is the displacement on ∂L, which is assumed to be differentiable on $\theta \in (\bar{\theta}(a), \bar{\theta}(\mathrm{a}) + 2\pi)$, but is otherwise arbitrary. In particular, if $\hat{\mathbf{u}}(\bar{\theta}(a)) \neq \hat{\mathbf{u}}(\bar{\theta}(a) + 2\pi)$, then the displacement suffers a jump $\mathbf{w}$ at the intersection of the new free surfaces. The function $P(r, \theta)$, in which the Heaviside step function $H(\cdot)$ appears, is intended to accommodate the displacement jump across non-straight free-surface profiles within L: if the free surfaces lie along a radial line in L, then $P = 0$.

The form (3) for the assumed displacement field in L is motivated as follows. First, as demanded by items 2 and 3 above, the first two terms represent a smooth (in fact, linear) variation with r between the (singular-valued) tip displacement $\mathbf{g}$ at $r = 0$, and the (arbitrary) boundary displacement at $r = a$. The linear variation seems to be consistent with the "as simple as possible" requirement (item 5). Also, the first term in $(3)_1$, with $P(r, \theta)$ as defined in $(3)_2$, leads to satisfaction of item 1. The last term in $(3)_1$ is motivated entirely by item 4: because it is continuous in both r and θ, and is zero at $r = 0$ and $r = a$, it has no bearing on items 1–3. However, the requirement that the traction $\mathbf{t}(\theta)$ and ∂L render the ER in overall force and moment equilibrium implies that the displacement in L cannot simply be chosen consistent with items 1–3, but otherwise arbitrarily. Specifically, the tip displacement $\mathbf{g}$, and the parameter v, must be chosen so that equilibrium is satisfied. The two components of $\mathbf{g}$ do not, by themselves, provide sufficient flexibility to guarantee satisfaction of all three equilibrium equations. Indeed, as will be seen in the subsequent development, the two force equilibrium equations lead directly to the determination of the two components of $\mathbf{g}$, with no appearance of the parameter v. Moment equilibrium, on the other hand, yields a simple expression for v, with no appearance of $\mathbf{g}$. This decoupling can be explained as follows: because the last term in $(3)_1$ involves the tangent vector $\mathbf{m}$, and is quadratic in r, it leads to a shear strain (and hence a shear traction) on ∂L. This shear traction produces zero force resultant on ∂L, but a nonvanishing net moment, which may be adjusted (through v) so that the moment equilibrium equation is satisfied.

Before proceeding with the development, it is worth emphasizing that the assumed displacement field (3) is not to be construed as an approximation for the displacement solution in a conventional boundary-value problem. Rather, it is a key ingredient in the construction of a generalized constitutive model for a material neighborhood that contains a separation front. Without it, it would not be possible to assign a displacement gradient on the ER boundary, nor, therefore, to connect the ER constitutive equation to that of the bulk continuum.

In the assumed displacement field (3), $\mathbf{g}$ is the displacement at the separation front, whereas v governs the twisting-type contribution. The nature of the assumed displacement field is illustrated in Fig. 2, in which the linear (with r) term, the twisting contribution, and the displacement jump across the new free surfaces are readily seen. The parameters $\mathbf{g}$ and v are to be determined from the requirement that contact forces acting across ∂L render the exclusion region in overall force and moment equilibrium. Preparatory to expressing this requirement, it is necessary to obtain the traction distribution on the bounding surface ∂L of the ER. The material response is assumed to be given by the isotropic, linearly-elastic relation

$$\mathbf{T} = \lambda \mathbf{1}\,\mathrm{tr}\,\mathbf{e} + 2\mu\,\mathbf{e}, \quad \mathbf{e} = 1/2(\nabla\mathbf{u} + \nabla\mathbf{u}^{\mathrm{T}}), \tag{4}$$

which applies everywhere in the body except in the ER, i.e., in the open region $B\backslash L = \{\mathbf{x} \mid \mathbf{x} \in B, |\mathbf{x} - \gamma| > a\}$. The local constitutive relation (4) also serves to define the traction distribution on ∂L as discussed above, with the strain, in turn, derived from the gradient of (3) evaluated at $r = a$. The result is

$$\begin{aligned}\mathbf{t}(\theta) = {}& \frac{\lambda}{a}\left[\left(\hat{\mathbf{u}} - \mathbf{g} + \frac{a\bar{\theta}'(a)}{2\pi}\mathbf{w}\right)\cdot\mathbf{n} + \hat{\mathbf{u}}'\cdot\mathbf{m}\right]\mathbf{n} + \\ & \frac{\mu}{a}\left[\hat{\mathbf{u}} - \mathbf{g} + \mathbf{n}(\hat{\mathbf{u}} - \mathbf{g})\cdot\mathbf{n} + (\hat{\mathbf{u}}'\cdot\mathbf{n} + v)\mathbf{m} + \frac{a\bar{\theta}'(a)}{2\pi}(\mathbf{w} + \mathbf{n}\,\mathbf{w}\cdot\mathbf{n})\right], \quad \bar{\theta}(a) \leqslant \theta < \bar{\theta}(a) + 2\pi.\end{aligned} \tag{5}$$

Here, a prime indicates derivative with respect to the argument. If the new free surfaces lie along a radial line within L, then all terms involving $\bar{\theta}'(a)$ are zero. Equation (5) gives the traction on the bounding surface ∂L in terms of the displacement there, and the elastic material constants. It is explicitly noted that the traction given by (5) suffers a jump where the new free surfaces meet ∂L; i.e., across $\theta = \bar{\theta}(a)^-$ and $\theta = \bar{\theta}(a)^+$.

With expression (5) in hand, the requirements of overall force and moment equilibrium for L are expressed by†

$$\int_{\bar{\theta}(a)}^{\bar{\theta}(a)+2\pi} \mathbf{t}\,\mathrm{d}\theta = \mathbf{0}, \quad \int_{\bar{\theta}(a)}^{\bar{\theta}(a)+2\pi} \mathbf{t}\cdot\mathbf{m}\,\mathrm{d}\theta = 0. \tag{6}$$

After substituting (5) into (6), and integrating by parts, there results

†The present development is being pursued with application to fracture-type problems in mind; i.e., for traction-free new-free-surfaces. Cutting and penetrating problems, wherein the new free surfaces may be subject to contact tractions, can be accommodated by including nonzero terms on the right-hand sides of (6).

$$(\lambda+3\mu)\pi\mathbf{g} = (2\lambda+3\mu)\int_{\bar\theta(a)}^{\bar\theta(a)+2\pi} \mathbf{n}\hat{\mathbf{u}}\cdot\mathbf{n}\,\mathrm{d}\theta - \lambda\int_{\bar\theta(a)}^{\bar\theta(a)+2\pi} \mathbf{m}\hat{\mathbf{u}}\cdot\mathbf{m}\,\mathrm{d}\theta$$
$$-\lambda\mathbf{N}\mathbf{w}\cdot\mathbf{M} - \mu\mathbf{M}\mathbf{w}\cdot\mathbf{N} + 1/2(\lambda+3\mu)a\bar\theta'(a)\mathbf{w}, \quad (7)$$

$$2\pi v = \mathbf{w}\cdot\mathbf{N}, \quad (8)$$

in which the definitions

$$\mathbf{N} \equiv \mathbf{n}(\bar\theta(a)), \quad \mathbf{M} \equiv \mathbf{m}(\bar\theta(a)), \quad (9)$$

have been introduced. Equations (7) and (8) determine the displacement $\mathbf{g}$ at Γ, and the twisting-type contribution v, as required by force and moment equilibrium of the exclusion region. As mentioned previously, eqns (6) determine $\mathbf{g}$ and v in a decoupled fashion: $\mathbf{g}$ is determined solely from force equilibrium, whereas v is a consequence of moment equilibrium only. Equations (7) and (8) illustrate that, under the assumption (3), the displacement field in L is completely determined by enforcement of overall force and moment equilibrium of L. Expressions analogous to (7) and (8) can be derived, although in rate form, in the context of inelasticity and/or finite deformations. Substitution of expressions (7) and (8) into (5) then yields the traction on ∂L as a functional of the boundary displacement $\hat{\mathbf{u}}$ on this surface. This is the desired result: a generalized constitutive model for the exclusion region, which yields an equilibrium traction distribution corresponding to an *arbitrary*, and not just linearly-varying, displacement distribution on ∂L. The displacement jump $\mathbf{w}$ across the new free surfaces makes an explicit appearance in this generalized constitutive model.

Conceptually, a parallel may be drawn between the foregoing development, and the theory of shells. In shell theory, a two-dimensional continuum theory is synthesized from the general three-dimensional theory by making a suitable assumption regarding the displacement of material points with respect to the mid-surface (or other suitable reference surface) of the shell. Under such an assumption, which is analogous to the ER assumed displacement field, pointwise satisfaction of equilibrium is not to be expected. However, the assumed kinematical field may be used in the local constitutive relations to render meaningful *overall* measures of force intensity, such as shear and normal forces and bending moments per unit length. These overall measures, analogous to the traction distribution $\mathbf{t}(\theta)$ on the ER boundary, are then used in suitable statements of overall equilibrium. The assumed displacement distribution through the thickness of the shell may be viewed as a means of facilitating a generalized constitutive model connecting, e.g., the rotation of line elements and the bending moment. This situation is strongly reminiscent of the relations between the displacement of, and the traction on, the ER boundary.

Before proceeding to the separation criterion, the consistency of the foregoing generalized constitutive model with the local material model is established. Specifically, the traction distribution on ∂L, given by (5) and with $\mathbf{g}$ and v obtained from (7) and (8), is sought for the special case of the surface displacement $\hat{\mathbf{u}}(\theta)$ given by a linear variation with spatial position. To this end, $\hat{\mathbf{u}}(\theta)$ and $\mathbf{w}$ (the displacement jump) are taken to be

$$\hat{\mathbf{u}}(\theta) = \mathbf{u}^0 + a\mathbf{H}\mathbf{n}(\theta) \quad \text{on } \partial L, \quad \mathbf{w} = \mathbf{0}, \quad (10)$$

wherein $\mathbf{u}^0$ is the displacement at the center γ of L, and $\mathbf{H}$ is the (constant) displacement gradient. At issue is the ability of the ER formulation to exactly represent the uniform deformation field

$$\mathbf{u}(\mathbf{x}) = \mathbf{u}^0 + \mathbf{H}(\mathbf{x}-\gamma) \quad (11)$$

from which (10) derives, as well as the traction distribution on ∂L associated with the corresponding uniform strain. Using expressions (10) in (7) and (8) results in

$$\mathbf{g} = \mathbf{u}^0, \quad v = 0, \tag{12}$$

from which is obtained (11) after introducing (12) into (3). Also, substituting (10) and (12) into (5) produces the traction $\mathbf{t}(\theta) = \mathbf{Tn}(\theta)$ on ∂L, where $\mathbf{T}$ is the (constant) stress deriving from the strain $\mathbf{e} = 1/2(\mathbf{H}+\mathbf{H}^T)$. It is therefore seen that the ER constitutive model is entirely consistent with the local constitutive description, in the absence of surface separation.

In the exclusion region theory, then, the boundary-value problem itself is modified by introducing a special constitutive characterization for a small, but finite, material neighborhood that contains the separation front. Elsewhere in the body, the usual local material model is presumed to apply. The motivation for introducing this special constitutive model lies in the fact that a material neighborhood containing a separation front, however small, does not, even approximately, suffer a linear displacement distribution on its boundary. Therefore, an enlarged constitutive characterization is required for some material neighborhood containing the separation front. The size of the exclusion region, given by the radius a, is material-dependent; its determination will be discussed in the next section. The derivation of the enlarged constitutive model has been carried out for the case of a linearly elastic, isotropic material. However, the exclusion region formulation is not restricted to this particular local material model; a similar development can be carried out for essentially any local constitutive model.

In the current linearly-elastic context, the material state is, of course, completely specified by the strain. For inelastic materials, the current state (and therefore the current stress) is a function of the deformation history of the material neighborhood. It is important to observe that, even for material points that have temporarily occupied the (moving) exclusion region, this deformation history is available as a consequence of the assumed displacement field (3). As well, and with reference to dynamic problems, the rate-of-change of momentum of the material within the ER is similarly available.

2.2. *Separation criterion*

To complete the picture, a constitutive postulate relating to the movement of the separation point Γ must be introduced. Consistent with the notion of introducing an enlarged constitutive model for the exclusion region, the occurrence of surface formation is assumed to be strongly correlated with the distortion of, and/or the tractions acting on, the boundary of the ER. Under this hypothesis, the details of the actual traction distribution within L are unknown but unimportant; a complete continuum-mechanical theory can be constructed by making reference to the traction on, and possibly the distortion of, ∂L only. This prescription is somewhat similar to, but more general than, stress-intensity-based crack-advance criteria of conventional fracture mechanics: there, the nature of the actual stress distribution very close to the crack tip is not known, but the crack is assumed to advance if a parameter that is characteristic of the stress-singularity intensity reaches a critical value. It is therefore evident that, in fracture-mechanics theory, the physically-required length scale is manifest in the crack-advance criterion only; whereas in the ER theory, it is present in the underlying boundary-value problem as well.

Because application of the separation criterion in the ER theory is not dependent upon an asymptotic solution, a broad range of phenomenologies may be represented. For example, in the case of ductile metals, transgranular fracture is often associated with the nucleation, growth, and coalescence of voids near the crack tip. Accordingly, the separation point might be made to advance in such a manner that a suitably-defined measure of the distortion of ∂L never exceeds a critical value. Similarly, metal-cutting processes, in which intense shearing deformation leads to new-free-surface formation, could be modeled in this manner. Brittle fracture, on the other hand, is perhaps better represented by a force-based criterion. Specifically, the separation point could be made to advance so that some scalar parameter that characterizes the forces acting on ∂L does not exceed a critical value. Consistent with the present focus on linearly elastic material behavior, this direction will be pursued here.

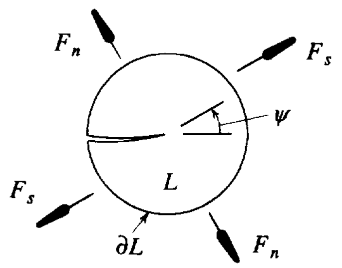

Fig. 3. Definition of the shear- and normal-opening forces on the exclusion region in relation to the candidate angle-of-advance ψ.

First, normal and shear forces acting on ∂L are defined in relation to a candidate direction-of-advance of Γ (Fig. 3). The candidate direction-of-advance is specified by the angle $\psi \in [\bar{\theta}(a), \bar{\theta}(a)+2\pi)$, whereas the actual direction-of-advance is given by $\psi = \bar{\psi}$, where $\bar{\psi}$ is to be determined. With reference to ψ, the normal force F_n and shear force F_s acting on a unit thickness of ∂L are given by

$$F_n = \int_{\bar{\theta}(a)}^{\bar{\theta}(a)+\psi} \mathbf{t}(\theta)\cdot\hat{\mathbf{m}}a\,\mathrm{d}\theta, \quad F_s = \int_{\bar{\theta}(a)}^{\bar{\theta}(a)+\psi} \mathbf{t}(\theta)\cdot\hat{\mathbf{n}}a\,\mathrm{d}\theta, \tag{13}$$

where the definitions $\hat{\mathbf{n}} \equiv (\cos\psi, \sin\psi)$ and $\hat{\mathbf{m}} \equiv (-\sin\psi, \cos\psi)$ have been used. The normal and shear forces acting on the part of ∂L that is complementary to that given by the range $\bar{\theta}(a) < \theta < \bar{\theta}(a)+\psi$ are, by equilibrium, equal in magnitude but opposite in sign to F_n and F_s. In the three-dimensional development, a third component of force acting along the tangent to Γ would be defined.

The forces F_n and F_s acting on the exclusion region are now used in a scalar *separation function*, which, by way of example, is here taken to be

$$\Phi = \frac{1}{a}\left[\langle F_n\rangle^2 + \beta F_s^2\right]^{1/2}. \tag{14}$$

Because F_n and F_s, as defined in (13), are actually forces per unit length of separation front, Φ has units of stress. In (14), $\langle\,\rangle$ are the Macaulay brackets, i.e., $\langle F_n\rangle = F_n$ if $F_n > 0$, otherwise $\langle F_n\rangle = 0$. The presence of the Macaulay brackets excludes the possibility of compressive (i.e. "crack-tip-closing") forces contributing to crack advance. Also, β is a material parameter that characterizes the relative sensitivity of the material to normal-opening ("mode I") vs shear-opening ("mode II") forces. It bears emphasis that Φ is a function of the candidate direction-of-advance angle ψ, due to the dependence on ψ of both F_n and F_s.

Surface separation is assumed to occur in such a manner that the separation function Φ satisfies the constraint

$$\Phi \leqslant \Phi_c, \tag{15}$$

wherein Φ_c is a material-dependent critical value. This separation criterion is applied as follows. First, the value $\psi = \bar{\psi}$ at which $\Phi(\psi)$ attains its maximum value is determined. The separation front will advance in this direction with such a speed $q \geqslant 0$ that (15) is maintained. In particular, three cases are possible:

$$\Phi(\bar{\psi}) < \Phi_c \Rightarrow q = 0 \tag{16a}$$

$$\Phi(\bar{\psi}) = \Phi_c: \quad \text{if } q = 0 \quad \text{implies } \frac{\mathrm{d}}{\mathrm{d}t}\left[\Phi(\bar{\psi})\right] \leqslant 0, \quad \text{then } q = 0 \tag{16b}$$

$$\text{if } q = 0 \quad \text{implies } \frac{\mathrm{d}}{\mathrm{d}t}\left[\Phi(\bar{\psi})\right] > 0, \quad \text{then } q > 0. \tag{16c}$$

If case (16c) holds, then q is determined by the condition that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\Phi(\bar{\psi})\right] = 0. \tag{17}$$

This prescription is analogous to the consistency condition of rate-independent plasticity theory, wherein the "plastic loading rate" is zero if the stress lies inside the yield surface, and is chosen so that the stress state remains on the yield surface otherwise. Accordingly, the condition for the determination of the speed q may be written in the compact form

$$(\Phi - \Phi_c)q = 0. \tag{18}$$

The direction-of-advance angle $\bar{\psi}$, i.e., the angle between $\dot{\gamma}$ and the x_1-direction, is determined by maximizing the function $\Phi(\psi)$. Local extrema of Φ occur at solutions of

$$F_s(\beta F_n - \langle F_n \rangle) - a\mathbf{t}(\psi) \cdot \left[\langle F_n \rangle \hat{\mathbf{m}} + \beta F_s \hat{\mathbf{n}}\right] = 0, \tag{19}$$

which results from setting the derivative of $\Phi(\psi)$ to zero. Because the left-hand side of (19) is a complicated function of ψ, a closed-form solution for $\bar{\psi}$ is not to be expected. In a computational setting, this expression is evaluated at every node on ∂L, with the root or roots then determined approximately. In the case of multiple roots, the separation function Φ itself is evaluated at each root to determine the absolute maximum. In any case, Φ must be evaluated at $\bar{\psi}$ in connection with enforcement of the condition (15).

The speed $q = |\dot{\gamma}|$ of the separation point, on the other hand, could be obtained by forming the time rate of Φ, and then applying (17). This calculation involves rates of the integrals in (13), wherein the lower bound of integration is itself, in general, a function of time. Although theorems are available for the rate of an integral taken over a moving surface (Gurtin *et al.*, 1989; Jaric, 1992), the resulting expression for q is prohibitively complicated. In any case, such an expression for q is of limited utility: in a computational setting, the constraint (15) is applied *incrementally*, with the incremental displacement of Γ determined so that (15) is satisfied at the end of each time step. This incremental displacement can be determined iteratively, without the need to form $\dot{\Phi}$. Alternatively, arbitrary increments of crack advance can be taken in the direction determined by (19), within certain limits imposed by considerations of accuracy. Then, the value of the global loading parameter is determined so that $\Phi = \Phi_c$ holds at the end of the load step. This determination can be made directly for proportional loading of elastic bodies; in other cases it must be determined iteratively.

Many modifications to, or replacements for, the simple separation function (14) can be envisaged. As mentioned previously, a separation function that involves the distortion of ∂L, instead of (or perhaps in addition to) the forces acting on ∂L, would perhaps be suitable for modeling ductile fracture of metals. However, such a development is most appropriately carried out in the context of large deformations and inelastic material response. Confining attention to brittle fracture, for which linear elasticity is adequate, many modifications to (14) may be introduced to model various physical phenomena. For example, anisotropic fracture resistance may be introduced by first modifying the vector $\mathbf{F} = (F_s, F_n)$ in relation to its orientation with respect to preferred material directions, before using it in the separation function. Heterogeneity, including the discrete case of layered media, can be modeled by a nonuniform distribution of the critical value Φ_c. In particular, a layered body would require a discontinuous distribution of Φ_c, with different values defined in each material, and distinct values on the interfaces between the materials. Finally, crack-tip frictional effects, in which crack-tip-closing forces tend to increase the resistance to mode-II-type fracture, could be addressed by making β a decreasing function of the compressive force $\langle -F_n \rangle$. In all cases, the separation criterion itself takes the form $\Phi \leqslant \Phi_c$, where Φ is fashioned to model the relevant physical phenomena.

2.3. *Summary of the theory and parameter determination*

The exclusion region theory involves formulation of a special constitutive model that governs the behavior of a small material neighborhood that contains the separation front. All material points outside this *exclusion region* are assumed to be subject to the specified local constitutive model. The ER constitutive model relates an *arbitrary* boundary displacement distribution to the corresponding traction distribution on the ER boundary. This enlarged constitutive characterization is consistent with, and in fact derives from, the local constitutive model itself. Indeed, it has been shown that, in the absence of surface separation, the ER constitutive model reduces to a version of the local constitutive model.

Ultimately, the purpose of introducing the exclusion region, along with its special constitutive characterization, is to formulate a predictive model for the advance of the separation front. This motivation parallels that of conventional fracture mechanics: there, the unbounded stress field ceases to be meaningful within some distance of the crack tip, at least in a strict sense. However, it *is* useful for the purpose of predicting the occurrence of crack advance. With respect to the ER theory, no claim is made that the traction distribution (5) is the one that actually exists at a radial distance a away from the crack tip. However, this traction distribution *is*, perhaps, useful for the purpose of predicting the occurrence of separation-front advance. However, unlike conventional fracture theory, the constitutive aspects of surface separation in the ER theory are not closely linked to an asymptotic solution, nor are they restricted by the particular local constitutive characterization used for the bulk continuum. In addition, the boundary-value problem solution under the ER theory involves only bounded field quantities, and is numerically convergent under mesh refinement.

It bears emphasis that the ER radius a is a material parameter, and in particular is not equivalent to the "plastic zone size" of conventional fracture mechanics. The latter is dependent upon the solution to the boundary-value problem itself, and in turn on the geometry and the loading, and therefore cannot be considered a material parameter. The exclusion region roughly corresponds to the somewhat vague concept of a "process zone" surrounding the crack tip. There is no reason why the exclusion region cannot be taken to have some other geometric form besides a circular tube of material. Indeed, the assumed configuration of L is precisely that—a constitutive assumption. Assuming, say, an elliptical cross-section would change the values of the empirical parameters, but would likely not change the predictive capability of the resulting theory in any significant manner. Also, such an assumption would clearly complicate both the generalized constitutive model for the ER, as well as the separation criterion. For these reasons, it does not seem fruitful to pursue such a generalization in the absence of any compelling reason to do so.

In the foregoing development of the exclusion region theory, the additional constitutive parameters introduced in connection with surface separation are a, β, and Φ_c. As will be discussed in the following section, $\sqrt{a}\Phi$ is a constant for a particular geometry and remote loading situation, whereas Φ scales linearly with the remote load. These simple relations hold as long as the local material response is linearly elastic, and if a is sufficiently small compared to the significant dimensions of the body (e.g., crack length). Accordingly, in the context of linear elasticity, a and the critical value Φ_c need not be measured independently; only the combination $\sqrt{a}\Phi_c$ must be correlated with experiment. In the case of symmetric mode-I-type loading, it turns out that (see below) $\sqrt{a}\Phi$ and K_I are essentially equal, leading to the simple relation $\sqrt{a}\Phi_c = K_{Ic}$. Remaining within the confines of linear elasticity, the parameter β is then to be determined from mixed-mode tests, in the same way that the effect of K_{II} on fracture initiation is measured.

With any departure from elastic material response, a and Φ_c become independently measurable parameters. Their individual roles can be illustrated by considering the case of a rate-independent elastic-plastic material. In this case, for a given geometry and loading

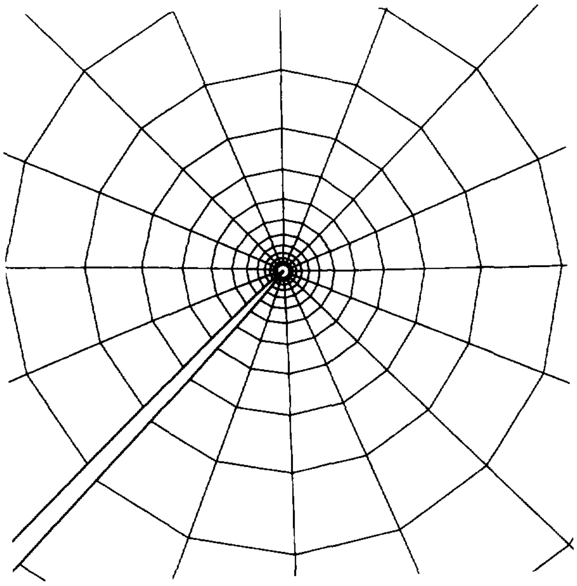

Fig. 4. Finite element mesh surrounding an exclusion region.

configuration, and for moderate load level, the exclusion region and all surrounding material would remain elastic, were the ER sufficiently large in size. In this event, the predictions of the ER theory would parallel those of LEFM, which is inappropriate if significant inelastic flow occurs. Reducing the size of the ER would cause the stress concentration in the vicinity of the ER to intensify, which, in turn, would lead to inelastic deformation of the surrounding material, as well as of the ER itself. With such a decrease in the size of the ER, the elastic relation $\sqrt{a}\Phi =$ constant for given loading ceases to hold, and in fact overpredicts the magnitude of the separation function Φ. It is therefore evident that Φ_c and a may be determined from measurement of the remote load at fracture initiation using two specimens of differing size. Measurements made using additional specimens then serve to assess the fidelity of the overall theroy. It bears emphasis, however, that the ER size, once determined for a particular material, does not vary with the extent of inelastic deformation in the crack tip vicinity. Depending on the crack size and the circumstances of loading, the ER may be embedded in a zone of substantial inelastic flow, or possibly none at all, when Φ reaches its critical value.

3. COMPUTATIONAL RESULTS

3.1. *Remarks on numerical implementation*

The ER theory as described above has been incorporated in a two-dimensional, quasistatic, linear finite element program. The code uses standard four-node bilinear quadrilateral elements, and can accommodate an arbitrary number of exclusion regions. The deformed mesh surrounding a typical exclusion region is shown in Fig. 4. Because of the integrals appearing in (7), the traction $\mathbf{t}(\theta)$ at a given point on ∂L depends on the displacement everywhere on ∂L. The negative of the traction, as given by (5), is applied as a natural boundary condition on the finite element mesh where it adjoins the ER. Nodal forces are thereby obtained for nodes lying on ∂L. These nodal forces, however, depend linearly on all the nodal displacements on ∂L. Accordingly, a fully-dense stiffness matrix is assembled that relates the nodal displacements to the nodal forces on ∂L. The computation of this matrix is based on eqns (5), (7), and (8).

The finite element code, called FEFRAC, uses a frontal solver (Irons, 1970; Hood, 1976) to solve the linear system of algebraic equations. However, element contributions are assembled into the active equation set in blocks, rather than on an element-by-element basis. Also, the factored equations are stored in a single, large vector in memory, rather than on disk, so that the number of reads and writes to disk is minimized or eliminated.

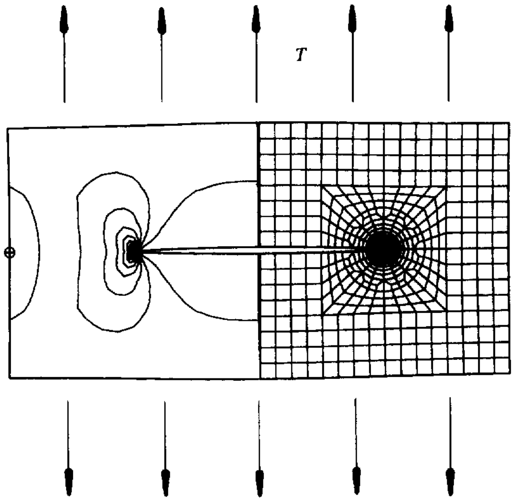

Fig. 5. Deformed mesh and contour plot of the vertical normal stress for the center-cracked specimen using the ER theory.

When this vector of factored equations reaches a user-specified maximum length, then the entire vector is written to disk. The presence of exclusion regions introduces nonstandard element-node dependencies into the frontal solution scheme. In particular, once an element with an ER-boundary node has been assembled, then all such elements for that ER must be assembled before any equations for the ER-node degrees-of-freedom can be eliminated. This is due to the fully-dense nature of the ER stiffness matrix. An element-ordering strategy has been devised that accommodates this feature.

FEFRAC returns the direction-of-advance angle $\bar{\psi}$ for each separation point, along with the value of the corresponding separation function in this direction. The angle $\bar{\psi}$ is found by evaluating the left-hand side of (19) at each node, and then searching for a root between each pair of nodes. The methodology that was developed to advance the separation point through the domain, called the "Arbitrary Local Mesh Replacement Method," is detailed by Rashid (1996). The ALMR method avoids the need to remesh the domain with each increment of crack advance. Also, it is worth noting that the ALMR method is not specific to the ER theory, and in fact may be used with essentially any numerical treatment of the crack tip.

3.2. *Example problem: correspondence between the ER theory and LEFM*

As mentioned previously, in the current context of linear elasticity, a direct correspondence can be established between conventional LEFM and the proposed theory. To this end, consider a rectangular region containing a central crack, and loaded by a vertical normal traction T. The deformed finite element mesh is shown in Fig. 5, along with a contour plot of the vertical normal stress. The half-length of the crack is denoted by b, and the overall dimensions of the region are $2b \times 4b$. Only one-half of the rectangular region was analyzed, due to symmetry. The finite element mesh consists of 623 nodes and 560 four-node bilinear elements. The linearly elastic analysis using FEFRAC required approximately 20 seconds of cpu time on a Sun Sparc 10 workstation. In the terminology of fracture mechanics, the crack tip is loaded in pure normal-opening mode (i.e., mode I). Accordingly, the exclusion region theory predicts that $\bar{\psi} = 0$, i.e., the direction-of-advance is horizontal.

Of particular interest here is the variation of the maximum of the separation function $\Phi_{\max} = \Phi(\bar{\psi})$ with the length scale b. In particular, $\Phi_{\max}$ is analogous to K_I in LEFM for this problem: both represent measures of a "crack tip driving force." The conventional theory predicts that K_I bears an inverse-square-root dependence on the length scale b, as indicated by the curve in Fig. 6. The symbols in the figure represent the critical applied traction T, normalized by Φ_c, at which the crack-advance criterion $\Phi(\bar{\psi}) = \Phi_c$ is reached. In Fig. 6, the length scale b, normalized by the ER radius a, is plotted along the horizontal

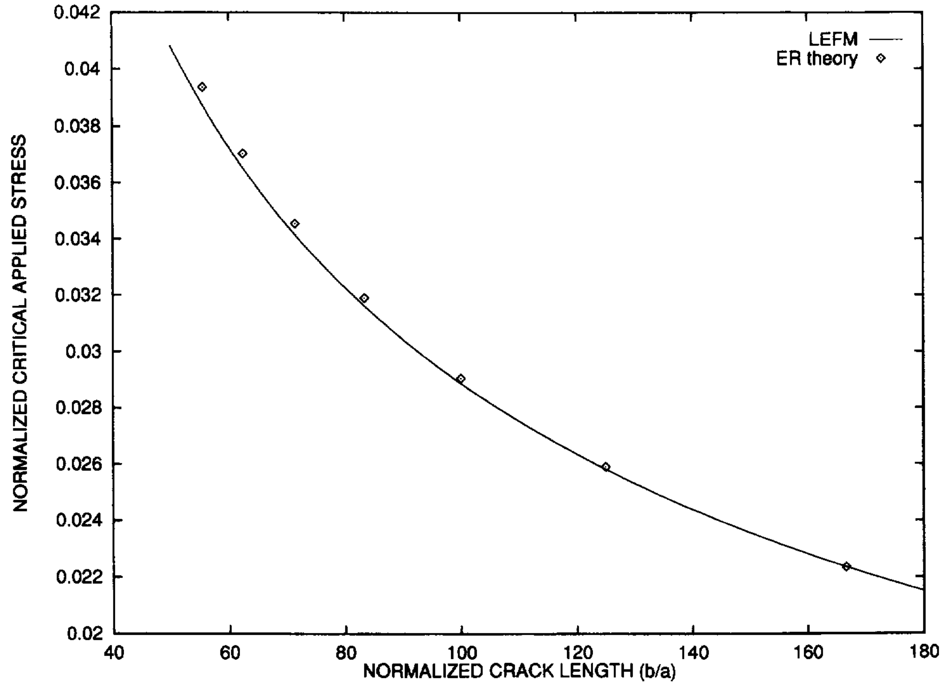

Fig. 6. Comparison of the effect of length scale on the critical applied stress between the ER theory (symbols) and LEFM. On the horizontal axis, the crack half-length b is normalized by the exclusion region radius, which is held constant. On the vertical axis, the critical applied traction T is normalized by the material parameter Φ_c.

axis. From the figure, it is evident that the proposed theory is capable of reproducing LEFM-type fracture behavior. As the ratio b/a decreases, i.e., as the exclusion region radius becomes larger in relation to the crack length, then a slight departure from the LEFM result is observed. This effect is due to the interaction of the material length scale with the length scale of the body.

In addition to the center-cracked panel, many other plane-strain mode-I configurations were analyzed, including single-edge-cracked strip, double-edge-cracked strip, and a center-cracked circular region with an applied K_I-field at the boundary. These analyses revealed a simple relation between the separation function Φ and the stress intensity factor:

$$K_I \approx \sqrt{a}\Phi. \tag{20}$$

The nature of the approximation in (20) is related to the smallness of the ER as compared to the crack length; typically, if $a < 0.03$ times the crack length, then eqn (20) holds to within one or two percent. Also, numerical experimentation has revealed that the right-hand side of (20) is actually a weak function of the Poisson ratio, varying a few percent as the Poisson ratio ranges from 0.0 to 0.4.

3.3. *Example problem: direction-of-advance*

To illustrate the determination of the direction of crack advance as predicted by the ER theory, the problem of a finite sheet with a central, diagonal crack, and loaded remotely by uniaxial tension, is considered. Erdogan and Shih (1963) measured the initial angle of crack advance in acrylic sheet for six initial crack angles ranging from 30° to 80° from the tensile axis (see also Shih, 1991, p. 13). The dimensions of the acrylic sheets were 9 in. × 18 in. × 0.1875 in., with a central crack of length 2 in. A typical finite element mesh used to analyze the problem is shown in Fig. 7. The mesh consists of 1600 elements and 1726 nodes, and the analyses each required approximately 85 seconds to carry out on a Sun Sparc 10 workstation. The initial angle of crack advance, as predicted by the ER theory, the minimum strain energy density criterion, and the maximum circumferential stress criterion, are compared to the experimental data in Fig. 7. The error bars in this figure represent the range

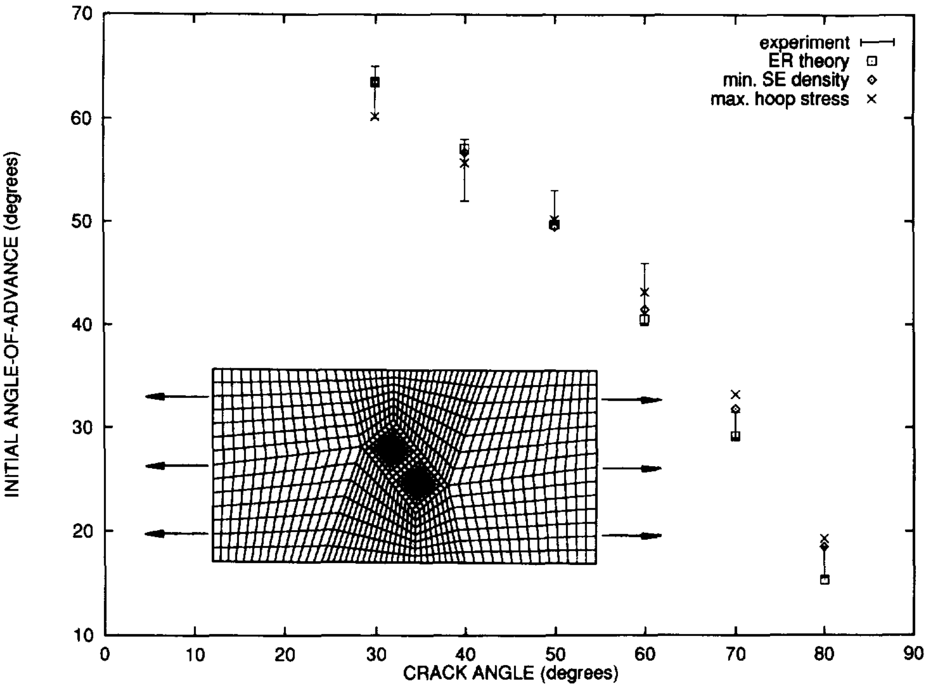

Fig. 7. Comparison of the initial angle-of-advance as predicted by the exclusion region theory, the minimum strain-energy density criterion, and the maximum hoop stress criterion. The bars indicate the range of the experimental values reported by Sih (1991). The crack angle is measured with respect to the loading axis, whereas the initial angle-of-advance is measured with respect to the existing crack. Inset shows typical mesh used in the ER-theory analysis.

of experimental data: Erdogan and Shih (1963) reported the initial angle-of-advance at both tips of the crack in four separate specimens for each of the crack angles $\phi = 30°$, $40°$, $50°$, $60°$, $70°$, and $80°$. With regard to the ER theory, the fit shown in Fig. 7 was obtained by setting the material parameter β to 0.9.

Data such as that of Fig. 7 can be used to refine the value of β, or indeed the postulated separation function (14) itself. In addition, measured values of not only initial angle-of-advance, but the critical applied load at crack initiation, could be used to guide the choice of the separation function. Currently, a strategy is being developed to determine the separation function, as a function of F_n and F_s, given data of this type.

4. CONCLUDING REMARKS

The proposed exclusion region theory is based upon the notion that surface separation proceeds whenever conditions within a small material neighborhood of the separation front reach a critical state. The mathematical statement of this critical condition takes the form of a scalar function Φ, defined at each point along the separation front, reaching a critical value Φ_c. Within this framework, broad latitude is available with regard to modeling physical phenomena that contribute to surface separation. Also, the direction-of-advance of the separation front naturally emerges as the direction in which Φ takes its maximum value. No additional considerations, beyond formulation of the separation function Φ, are required to determine the direction-of-advance.

Of course, the microscopic processes attending surface separation vary widely among materials and circumstances. Intergranular cleavage of ceramics, for example, would seem to share little in common with crazing and rupture in polymers. However, the present proposal represents an attempt to provide a common continuum-mechanical framework within which a wide variety of such processes can be modeled for engineering purposes. The common feature of all surface-separation processes is the existence of a material neighborhood which does not, even approximately, suffer a linear displacement field on its boundary. The ER theory embraces an enlarged conception of a constitutive model, within which the boundary traction–boundary displacement relationship may be cast under such circumstances. The advance of the separation front is governed by a criterion that arises naturally in connection with the generalized ER constitutive model.

Formulation of the generalized ER constitutive model relies heavily upon the assumed displacement field (3), from which derives the deformation gradient, and, in turn, the stress and traction on the boundary of the ER. No claim is made that the assumed displacement field within the exclusion region approximates the actual one. Rather, the ER displacement field is simply a means of formulating a generalized constitutive model for the ER, consistent with the local constitutive characterization that applies in the bulk continuum. In this connection, the unbounded stress fields typically appearing in the conventional fracture theory are perhaps no more physically realistic, in a strict sense, than the ER displacement field. Both, however, are useful for making rational predictions of crack advance. The proposed theory may offer certain advantages, such as a decoupling of the separation criterion from the bulk constitutive model, and a convergent numerical implementation.

The exclusion region theory offers a rather convenient numerical implementation in a finite element setting. Specifically, each exclusion region represents a compliant inclusion in the finite element mesh. The exclusion region stiffness matrix is readily obtained from equations of the type (5), (7) and (8). The resulting finite element method is convergent with respect to mesh refinement, due to the finite length scale introduced with the ER radius. Accordingly, the stress and strain fields remain bounded in the presence of a separation front. As illustrated by the example problem in Section 3.2, the ER theory is capable of representing an inverse-square-root dependence of critical load on crack length, as predicted by LEFM. The modeling capabilities of the proposed theory with regard to inelastic fracture, and to cutting, tearing, and penetration processes, are currently under investigation.

Acknowledgements—It is a pleasure to acknowledge the support of Sandia National Laboratories under contract AJ-3278, and the National Science Foundation under grant CMS-9522730.

REFERENCES

Barenblatt, G. I. (1962). The mathematical theory of equilibrium cracks in brittle fracture. *Advances in Applied Mechanics* **7,** 55–129.

Dugdale, D. A. (1960). Yielding of steel sheets containing slits. *Journal of Mechanics, Physics and Solids* **8,** 100–104.

Duva, J. M. (1990). The singularity strength at the apex of a wedge undergoing finite deformations. *Journal of Applied Mechanics* **57,** 577–580.

Erdogan, F. and Sih, G. C. (1963). On the crack extension in plates under plane loading and transverse shear. *Journal of Basic Engineering* **85,** 519–527.

Eshelby, J. D. (1956). The continuum theory of lattice defects. In *Solid State Physics*, vol. III (eds F. Seitz and D. Turnbull), Academic Press.

Fager, L.-O., Bassani, J. L., Hui, C.-Y. and Xu, D.-B. (1991). Aspects of cohesive zone models and crack growth in rate-dependent materials. *International Journal of Fracture* **52,** 119–144.

Freund, L. B. (1978). Stress intensity factor calculations based on a conservation integral. *International Journal of Solids and Structures* **14,** 241–250.

Gurtin, M. E., Struthers, A. and Williams, W. O. (1989). A transport theorem for moving interfaces. *Quarterly Applied Mathematics* **47,** 773–777.

Herrmann, A. G. and Herrmann, G. (1981) On energy-release rates for a plane crack. *Journal of Applied Mechanics* **48,** 525–528.

Hillerborg, A., Modeer, M. and Petersson, P.-E. (1976). Analysis of crack formation and crack growth in concrete by means of fracture mechanics and finite elements. *Cement and Concrete Research* **6,** 773–782.

Hood, P. (1976). Frontal solution program for unsymmetric matrices. *International Journal of Numerical Methods in Engineering* **10,** 379–399.

Hutchinson, J. W. (1968). Singular behavior at the end of a tensile crack tip in a hardening material. *Journal of Mechanics, Physics and Solids* **16,** 13–31.

Irons, B. M. (1970). A frontal solution program for finite element analysis. *International Journal of Numerical Methods in Engineering* **2,** 5–32.

Irwin, G. R. (1957). Analysis of stresses and strains near the end of a crack traversing a plate. *Journal of Applied Mechanics* **24,** 361–364.

Jaric, J. P. (1992). On a transport theorem for moving interface. *International Journal of Engineering Science* **30,** 1535–1542.

Knauss, W. G. and Losi, G. U. (1993). Crack propagation in a nonlinearly viscoelastic solid with relevance to adhesive bond failure. *Journal of Applied Mechanics* **60,** 793–801.

Knowles, J. K. and Sternberg, E. (1973). An asymptotic finite-deformation analysis of the elastostatic field near the tip of a crack. *Journal of Elasticity* **3**, 67 107.

Liaw, B. M., Jeang, F. L., Hawkins, N. M. and Kobayashi, A. S. (1990) Fracture-process zone for mixed-mode loading of concrete. *Journal of Engineering Mechanics* **116**, 1560–1579.

Liebowitz, H., Sandhu, J. S., Lee, J. D. and Menandro, F. C. M. (1995). Computational fracture mechanics: research and application. *Engineering Fracture Mechanics* **50**, 653–670.

Liu, H. W., Kunio, T., Weiss, V. and Okamura, H. (eds) (1991). *Fracture Mechanics of Ductile and Tough Materials and its Applications to Energy Related Structures*, Proceedings of the USA-Japan joint seminar held at Hyama, Japan, 12–16 November 1979, Martinus Nijhoff, The Hague.

Moran, B. and Shih, C. F. (1987). A general treatment of crack tip contour integrals. *International Journal of Fracture* **35**, 295–310.

Rashid, M. M. (1996). The arbitrary local mesh replacement method: an alternative to remeshing for crack propagation analysis. (submitted).

Rashid, M. M. (1995). Computational simulation of surface formation in solid continua. In *Proc.* 1995 *Conf. on Computational Engineering Science* (eds S. N. Atluri, G. Yagawa and T. A. Cruse,), Mauna Lani, Hawaii, 30 July–3 August, pp. 1709–1714.

Rice, J. R. (1968). A path-independent integral and the approximate analysis of strain concentration by notches and cracks. *Journal of Applied Mechanics* **35**, 379–386.

Rice, J. R. and Rosengren, G. F. (1968). Plane strain deformation near a crack tip in a power-law hardening material. *Journal of Mechanics, Physics and Solids* **16**, 1–12.

Sharma, S. M., Srivastava, A. K. and Aravas, N. (1994). Higher order crack tip asymptotic solutions for quasi-static steady crack growth in nonlinear viscous solids. *Journal of Mechanics, Physics and Solids* **42**, 159–179.

Sih, G. C. (1991). *Mechanics of Fracture Initiation and Propagation*, Kluwer Academic Publishers, Dordrecht.

Tvergaard, V. and Hutchinson, J. W. (1992). The relation between crack growth resistance and fracture process parameters in elastic-plastic solids. *Journal of Mechanics, Physics and Solids* **40**, 1377–1397.

Willis, J. R. (1967). Crack Propagation in viscoelastic solids. *Journal of Mechanics, Physics and Solids* **15**, 229–240.

Xia, L. and Shih, C. F. (1995). Ductile crack growth—I. A numerical study using computational cells with microstructurally-based length scales. *Journal of Mechanics, Physics and Solids* **43**, 233–259.

Xia, L., Shih, C. F. and Hutchinson, J. W. (1995). A computational approach to ductile crack growth under large scale yielding conditions. *Journal of Mechanics, Physics and Solids* **43**, 389–413.